Replica symmetry breaking and the renormalization group theory of the weakly disordered ferromagnet

1995 J. Phys. A: Math. Gen. 28 5183

(http://iopscience.iop.org/0305-4470/28/18/010)

J. Phys. A: Math. Gen. 28 (1995) 5183–5206. Printed in the UK

Replica symmetry breaking and the renormalization group theory of the weakly disordered ferromagnet

Viktor Dotsenko†‡ and D E Feldman†

† Landau Institute for Theoretical Physics, Russian Academy of Sciences, Kosygina 2, 117940 Moscow, Russia

‡ Laboratoire de Physique Theorique, Ecole Normale Superieure, 24 rue Lhomond, 75231 Paris, France

Received 22 February 1995, in final form 15 June 1995

Abstract. We study the critical properties of the weakly disordered p-component ferromagnet in terms of the renormalization group (RG) theory generalized to take into account the replica symmetry breaking (RSB) effects coming from the multiple local minima solutions of the mean-field equations. Recently it has been shown that for $p < 4$ the traditional RG flows at dimensions $D = 4 - \epsilon$, which are usually considered as describing the disorder-induced universal critical behaviour, are unstable with respect to the RSB potentials as found in spin glasses, and a new type of stable one-step RSB fixed point has been discovered [1]. Here it is demonstrated that for a general type of the Parisi RSB structures there exist no stable fixed points, and the RG flows lead to the *strong-coupling regime* at the finite scale $R_* \sim \exp(1/u)$, where u is the small parameter describing the disorder. The physical consequences of the obtained RG solutions are discussed. In particular, we argue that discovered RSB strong-coupling phenomena indicate the onset of a new spin-glass-type critical behaviour in the temperature interval $\tau < \tau_* \sim \exp(-1/u)$ near Tr. The possible relevance of the considered RSB effects for the Griffith phase is also discussed.

1. Introduction

In this paper we study the effects produced by weak quenched disorder on the critical phenomena in the ferromagnetic spin systems near the phase transition point. According to the usual fluctuation theory of the second-order phase transitions the leading singularities of the thermodynamical functions near the critical temperature T_c are fully described in terms of the set of universal critical exponents [2]. The only relevant spatial scale in the critical region is the correlation length R_c which scales as $\sim \tau^{-\nu}$, where $\tau \equiv (T/T_c - 1) \ll 1$ is the reduced temperature parameter and ν is the correlation length critical exponent.

According to the traditional point of view worked out many years ago [3–5], the effect produced by weak disorder on the critical behaviour remains negligible so long as the correlation length R_c is not too large, i.e. for temperatures T not too close to T_c. However, in the close vicinity of the critical point, where the effective strength of the disorder measured with respect to the correlation length becomes non-small, one has no grounds, in general, for believing that the effect of the disorder will be small. According to the so-called Harris criterion [3], weak quenched disorder strongly affects the critical behaviour only if α, the specific heat exponent of the pure system, is positive (i.e. the specific heat of the pure system is divergent at the critical point). In this case a new universal critical behaviour, with new critical exponents, is established sufficiently close to the phase transition point for $\tau \ll \tau_u \equiv u^{1/\alpha}$ [4, 5]. In contrast, when $\alpha < 0$ (the specific heat is finite), the disorder appears to be irrelevant for the critical behaviour.

Let us consider the usual D-dimensional scalar field Ginsburg–Landau Hamiltonian with a double-well potential:

$$H = \int \mathrm{d}^D x \left[\tfrac{1}{2}(\nabla\phi(x))^2 + \tfrac{1}{2}[\tau - \delta\tau(x)]\phi^2(x) + \tfrac{1}{4}g\phi^4(x)\right]. \tag{1.1}$$

Here $\tau = (T/T_c - 1) \ll 1$, and quenched disorder is introduced by random fluctuations of the effective transition temperature $\delta\tau(x)$. The probability distribution for $\delta\tau(x)$ is taken to be symmetric and Gaussian:

$$P[\delta\tau] = p_0 \exp\left(-\frac{1}{4u}\int \mathrm{d}^D x\, (\delta\tau(x))^2\right) \tag{1.2}$$

where $u \ll 1$ is the small parameter which describes the strength of the disorder, and p_0 is the normalization constant.

In general terms, to derive the critical properties of such a system, one has to integrate over all local field configurations up to the scale of the correlation length. This type of calculation is usually performed using a renormalization group (RG) scheme, which self-consistently takes into account all the fluctuations of the field on length scales up to R_c. In actual calculations the traditional results for the critical properties of the system under consideration are obtained in terms of the standard RG procedure developed for dimensions $D = 4 - \epsilon$, where $\epsilon \ll 1$. Then one finds that in the presence of the quenched disorder the pure system fixed point becomes unstable, and the RG rescaling trajectories arrive at another (universal) fixed point $g_* \neq 0$; $u_* \neq 0$, which yields the new critical exponents describing the critical properties of the system with disorder.

However, there exists an important point which is missing in the traditional approach. Consider the ground-state properties of the system described by the Hamiltonian (1.1). Configurations of the fields $\phi(x)$ which correspond to local minima in H satisfy the saddle-point equation

$$-\Delta\phi(x) + (\tau - \delta\tau(x))\phi(x) + g\phi^3(x) = 0. \tag{1.3}$$

Clearly, the solutions of these equations depend on a particular configuration of the function $\delta\tau(x)$ being inhomogeneous. In such solutions the non-zero values of $\phi(x)$ exist in regions of space where $(\tau - \delta\tau(x)) < 0$. Moreover, one finds a *macroscopic* number of local minima solutions of the saddle-point equation (1.3). Indeed, for a given realization of the random function $\delta\tau(x)$ there exists a macroscopic number of spatial 'islands' where $(\tau - \delta\tau(x))$ is negative (so that the local effective temperature is below T_c), and in each of these 'islands' one finds two local minimum configurations of the field: one which is 'up', and another which is 'down'. These local minimal energy configurations are separated by finite energy barriers, whose heights become larger as the size of the 'islands' is increased.

The problem is that the traditional RG approach is only a perturbative theory in which one integrates over the deviations of the field around the ground-state configuration, and it cannot take into account other local minimum configurations which are 'beyond barriers'. This problem does not arise in the pure systems, where the solution of the saddle-point equation is unique. However, in the presence of quenched disorder, when one gets numerous local minimum configurations separated by finite barriers, the direct application of the traditional RG scheme may be questioned. In this situation a systematic approach must consist of both integration (in an RG way) over fluctuations around the local minima configurations, and summation over all these local minima. In view of the fact that the local minima configurations are defined by the random quenched function $\delta\tau(x)$ in an essentially non-local way, the possibility of implementing successfully such a systematic approach seems rather hopeless.

On the other hand there exists another technique which has been developed specifically for dealing with systems which exhibit numerous local minima states. It is the Parisi replica symmetry breaking (RSB) scheme which has proved to be crucial in the mean-field theory of spin glasses (see e.g. [6]). Recent studies show that in certain cases the RSB approach can also be generalized for situations where one has to deal with fluctuations as well [7–9]. Moreover, recently it has been shown that the RSB technique can be successfully applied for the RG studies of the critical phenomena in the sine–Gordon model where remarkable instability of the RG flows with respect to the RSB modes has been discovered [10].

In [1] qualitative arguments were presented showing that the summation over multiple local minima configurations in the p-component spin system could provide additional non-trivial RSB interaction potentials for the fluctuating fields [1]. The idea was that hopefully after summation over these discrete degrees of freedom the critical phenomena could then be studied in terms of the usual RG scheme for the fluctuating fields with modified effective interaction potentials. It is believed that this type of generalized RG scheme self-consistently takes into account relevant degrees of freedom coming from the numerous local minima. In particular, the instability of the traditional replica symmetric (RS) fixed points with respect to the RSB would indicate that the multiplicity of the local minima can be relevant for the critical properties in the fluctuation region. In [1] due to several simplifying assumptions, the effective ϕ^4 interaction potentials have been calculated explicitly, and it was demonstrated that the structure of replica interactions belongs to the so-called one-step RSB class. It was also shown that, whenever the disorder is relevant for the critical behaviour (for the number of spin components $p < 4$), the usual RS fixed points (which are usually considered as describing the disorder-induced universal critical behaviour) are unstable with respect to 'turning on' the RSB potential, and a new type of stable one-step RSB fixed point has been discovered. These results made it possible to calculate the corresponding critical exponents, which (unlike the traditional RS situation) appeared to be *non-universal*, being dependent on the concrete statistical properties of the disorder.

The important point, however, is that the non-trivial fixed points obtained in [1] can be shown to be stable only within the one-step RSB subspace, being unstable with respect to continuous RSB modes. Until it is certain that the starting ϕ^4 potentials have strictly one-step RSB structure, this type of instability is not important (just like the instability with respect to the RSB is irrelevant until the starting ϕ^4 interactions are strictly RS). However, at the present stage one cannot be sure that considered systems are indeed adequately described by the one-step RSB ansatz. Moreover, according to [1], in the most interesting Ising case ($p = 1$) there exist no stable fixed points within one-step RSB subspace at all.

In this paper we are going to study the critical properties of weakly disordered p-component systems (including the $p = 1$ case) along the same lines as the generalized RG approach taking into account the possibility of a *general type* of the RSB potentials for the fluctuating fields. It will be shown in section 2 that in this case the RG flows does not arrive at any fixed point, and instead they actually lead to the *strong-coupling regime* at the finite spatial scale $R_* \sim \exp(1/u)$ (which corresponds to the temperature scale $\tau_* \sim \exp(-1/u)$). For this scale the structure of the renormalized interactions develops strong RSB, and the values of the corresponding interaction parameters are getting non-small.

Usually the strong-coupling asymptotics of RG flows indicate that certain essentially non-perturbative excitations have to be taken into account. Presumably, in the present model these are due to exponentially rare 'instantons' in the spatial regions, where the value of

$\delta\tau(x) \sim 1$, and correspondingly the local values of the field $\varphi(x)$, must be $\sim \pm 1$. (A distant analogue of this situation exists in the two-dimensional Heisenberg model where the Poliakov renormalization develops into the strong-coupling regime at a finite (exponentially large) scale which is known to be due to the nonlinear localized instanton solutions [12].)

In section 3 we derive the physical consequences of the obtained RG solutions for observable quantities. In particular we show that due to the absence of fixed points at the disorder dominated scales $R \gg u^{-\nu/\alpha}$ (or at the corresponding temperature scales $\tau \ll u^{1/\alpha}$) there must be no simple scaling behaviour of the physical quantities. Besides, we demonstrate that the replica structure of the SG-type two-point correlation functions is characterized by the strong RSB.

In section 4 we consider the special case of systems with the number of spin components $p = 4$, in which the pure system specific heat critical exponent $\alpha = 0$. Here the disorder appears to be marginally irrelevant in a sense that it does not change the critical exponents. Nevertheless, the critical behaviour itself (described in terms of the logarithmic singularities) is effected by the disorder, and moreover, the RSB phenomena is demonstrated to be relevant in this case as well.

Finally, in section 5 we discuss possible physical interpretations of the obtained results. *In particular, we speculate that the explicit* RSB *structure of the* SG*-type two-point correlation* functions could be interpreted as indicating the onset of a new type of critical behaviour of SG nature. We also briefly discuss the possible relevance of the considered RSB phenomena for the Griffith phase which is known to exist in a finite temperature interval near T_c [13].

2. RSB in the renormalization group theory

We consider the p-component ferromagnet with quenched random effective temperature fluctuations, which near the transition point can be described by the usual Ginzburg–Landau Hamiltonian

$$H[\delta\tau, \phi] = \int \mathrm{d}^D x \left[\tfrac{1}{2} \sum_{i=1}^{p} (\nabla\phi_i(x))^2 + \tfrac{1}{2}(\tau - \delta\tau(x)) \sum_{i=1}^{p} \phi_i^2(x) + \tfrac{1}{4} g \sum_{i,j=1}^{p} \phi_i^2(x)\phi_j^2(x) \right] \tag{2.1}$$

where the quenched random temperature $\delta\tau(x)$ is described by the Gaussian distribution (1.2).

To carry out the appropriate average over quenched disorder we can use the standard replica approach, in which one averages the nth ($n \to 0$) power of the partition function, $Z_n \equiv \overline{Z^n[\delta\tau]}$, where $\overline{(\ldots)}$ denotes the averaging over $\delta\tau(x)$ with the probability distribution (1.2). Simple integration yields

$$\begin{aligned} Z_n \equiv \overline{Z^n(\delta\tau)} = \int D\phi_i^a(x) \exp\Bigg[- \int \mathrm{d}^D x \Bigg(\tfrac{1}{2} \sum_{i=1}^{p} \sum_{a=1}^{n} [\nabla\phi_i^a(x)]^2 \\ + \tfrac{1}{2}\tau \sum_{i=1}^{p} \sum_{a=1}^{n} [\phi_i^a(x)]^2 + \tfrac{1}{4} \sum_{i,j=1}^{p} \sum_{a,b=1}^{n} g_{ab} [\phi_i^a(x)]^2 [\phi_j^b(x)]^2 \Bigg) \Bigg] \end{aligned} \tag{2.2}$$

where

$$g_{ab} = g d_{ab} - u\,. \tag{2.3}$$

To study the critical properties of this system we use the standard RG procedure developed for dimensions $D = 4 - \epsilon$, where $\epsilon \ll 1$. Along the lines of the usual rescaling scheme (see e.g. [2]) one easily gets the following (one-loop) RG equations for the interaction parameters g_{ab}:

$$\frac{\mathrm{d}g_{ab}}{\mathrm{d}\xi} = \epsilon g_{ab} - \frac{1}{8\pi^2}\left(4g_{ab}^2 + 2(g_{aa} + g_{bb})g_{ab} + p\sum_{c=1}^{n} g_{ac}g_{cb}\right) \tag{2.4}$$

where ξ is the standard rescaling parameter. Note that the above equations are valid for *arbitrary* structure of the interaction matrix g_{ab}, and not only for the RS type (2.3).

Changing $g_{ab} \to 8\pi^2 g_{ab}$, and $g_{a\neq b} \to -g_{a\neq b}$ (so that the off-diagonal elements would be positively defined), and introducing $\tilde{g} \equiv g_{aa}$, we get the following RG equations:

$$\frac{\mathrm{d}g_{ab}}{\mathrm{d}\xi} = \epsilon g_{ab} - (4+2p)\tilde{g}g_{ab} + 4g_{ab}^2 + p\sum_{c\neq a,b}^{n} g_{ac}g_{cb} \qquad (a \neq b) \tag{2.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{g} = \epsilon\tilde{g} - (8+p)\tilde{g}^2 - p\sum_{c\neq 1}^{n} g_{1c}^2. \tag{2.6}$$

If one takes the matrix g_{ab} to be RS, as in the starting form of equation (2.3), then one would recover the usual RG equations for the parameters g and u, and eventually one would obtain the well known results for the fixed points and the critical exponents [4, 5]. Here we leave apart the question as to how perturbations out of the RS subspace could arise (see discussion in [1]) and formally consider the RG equations (2.5), (2.6) assuming that the matrix g_{ab} has a general Parisi RSB structure.

According to the standard technique of the Parisi RSB algebra (see e.g. [6]), in the limit $n \to 0$ the matrix g_{ab} is parametrized in terms of its diagonal elements $\tilde{g}$ and the off-diagonal *function* $g(x)$ defined in the interval $0 < x < 1$. All the operations with the matrices in this algebra can be performed according to the following simple rules (see e.g. [7, 14]):

$$g_{ab}^k \to (\tilde{g}^k; g^k(x)) \tag{2.7}$$

$$(\hat{g}^2)_{ab} \equiv \sum_{c=1}^{n} g_{ac}g_{cb} \to (\tilde{c}; c(x)) \tag{2.8}$$

where

$$\begin{aligned}\tilde{c} &= \tilde{g}^2 - \int_0^1 \mathrm{d}x\, g^2(x)\\ c(x) &= 2\left(\tilde{g} - \int_0^1 \mathrm{d}y\, g(y)\right)g(x) - \int_0^x \mathrm{d}y[g(x) - g(y)]^2.\end{aligned} \tag{2.9}$$

The RS situation corresponds to the case $g(x) = \text{constant}$, independent of x.

Using the above rules from (2.5) and (2.6) one gets

$$\frac{\mathrm{d}}{\mathrm{d}\xi}g(x) = (\epsilon - (4+2p)\tilde{g})g(x) + 4g^2(x) - 2pg(x)\int_0^1 \mathrm{d}y\, g(y) - p\int_0^x \mathrm{d}y\,(g(x) - g(y))^2 \tag{2.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{g} = \epsilon\tilde{g} - (8+p)\tilde{g}^2 + p\overline{g^2} \tag{2.11}$$

where $\overline{g^2} \equiv \int_0^1 \mathrm{d}x\, g^2(x)$.

Usually in the studies of the critical behaviour one is looking for the stable fixed-point solutions of the RG equations. From equation (2.10) one can easily find out what the structure of the function $g(x)$ at the fixed point, $\frac{d}{d\xi}g(x) = 0$, $\frac{d}{d\xi}\tilde{g} = 0$ should be. Taking the derivative over x twice, one gets, from equation (2.10), $g'(x) = 0$. This means that either the function $g(x)$ is constant (which is the RS situation), or it has a step-like structure. It is interesting to note that the structure of fixed-point equations is similar to that for the Parisi function $q(x)$ near T_c in the Potts spin glasses [15], and it is the term $g^2(x)$ in equation (2.10) which is known to produce the one-step RSB solution there. The numerical solution of the above RG equations convincingly demonstrates that whenever the trial function $g(x)$ has the many-step RSB structure, it quickly develops into the one-step one with the coordinate of the step being the most right step of the original many-step function.

In [1] the following one-step RSB ansatz for the function $g(x)$ has been considered:

$$g(x) = \begin{cases} g_0 & \text{for } 0 \leqslant x < x_0 \\ g_1 & \text{for } x_0 < x \leqslant 1 \end{cases} \tag{2.12}$$

where $0 \leqslant x_0 \leqslant 1$ is the coordinate of the step.

In terms of this ansatz the above fixed-point equations have several non-trivial solutions.

(i) The RS fixed point which corresponds to the pure system:

$$g_0 = g_1 = 0 \qquad \tilde{g} = \frac{1}{8+p}\epsilon. \tag{2.13}$$

This fixed point (in accordance with the Harris criterion) is stable for the number of spin components $p > 4$, and becomes unstable for $p < 4$.

(ii) The disorder-induced RS fixed point (for $p > 1$) [4, 5]:

$$g_0 = g_1 = \epsilon\frac{4-p}{16(p-1)} \qquad \tilde{g} = \epsilon\frac{p}{16(p-1)}. \tag{2.14}$$

It was usually considered to be the solution which describes the new universal critical behaviour in systems with impurities. This fixed point has been shown to be stable (with respect to the RS deviations!) for $p < 4$, which is consistent with the Harris criterion. (For $p = 1$ this fixed point involves an expansion in powers of $(\epsilon)^{1/2}$ and this structure is only revealed within a two-loop approximation.) However, the stability analysis with respect to the RSB deviations shows that this fixed point is *always unstable* [1]. Therefore, whenever the disorder is relevant for the critical behaviour, the RSB perturbations must be becoming the dominant factor in the asymptotic large-scale limit.

(iii) The one-step RSB fixed point [1]:

$$g_0 = 0 \qquad g_1 = \epsilon\frac{4-p}{16(p-1) - px_0(8+p)} \qquad \tilde{g} = \epsilon\frac{p(1-x_0)}{16(p-1) - px_0(8+p)}. \tag{2.15}$$

This fixed point can be shown to be stable (within the one-step RSB subspace!) for

$$1 < p < 4 \qquad 0 < x_0 < x_c(p) \equiv \frac{16(p-1)}{p(8+p)}. \tag{2.16}$$

In particular, $x_c(p=2) = \frac{4}{5}$; $x_c(p=3) = \frac{32}{33}$, and $x_c(p=4) = 1$. Using the result (2.15) one can easily obtain the corresponding critical exponents which become non-universal, being dependent on the starting parameter x_0. In particular, for the critical exponents of the correlation length and the specific heat one finds [1]

$$\nu(x_0) = \frac{1}{2} + \frac{1}{2}\epsilon\frac{3p(1-x_0)}{16(p-1) - px_0(p+8)} \tag{2.17}$$

$$\alpha(x_0) = -\frac{1}{2}\epsilon\frac{(4-p)(4-px_0)}{16(p-1) - px_0(p+8)}. \tag{2.18}$$

Unfortunately, to obtain the first correction to the critical exponent η of the two-point correlation function one has to study the RG fixed points in the next order ($\sim\epsilon^2$) approximation. Technically such types of calculations are much more cumbersome and the result for the critical exponent η still remains to be found.

The problem, however, is that if the parameter x_0 of the starting function $g(x; \xi = 0)$ (or, more generally, the coordinate of the most right step of the many-step starting function) is beyond the 'stability interval', (2.16), such that $x_c(p) < x_0 < 1$, then there exist *no stable fixed points* of the RG equations (2.10), (2.11). One faces the same situation, of course, in the case of a general continuous starting function $g(x; \xi = 0)$. Moreover, according to (2.16) there exist no stable fixed points out of the RS subspace in the most interesting Ising case, $p = 1$.

It should be stressed that unlike the RS situation for $p = 1$, where one finds the stable $\sim\sqrt{\epsilon}$ fixed point in the two-loop RG equations [5], here adding next-order terms in the RG equations does not cure the problem. In the considered RSB case one finds that in the two-loop RG equations the values of the parameters in the fixed point are formally becoming of the order of one, and it signals that we are entering the strong-coupling regime where all the orders of the RG become relevant.

Nevertheless, to get at least some information about the physics behind this instability phenomena, one can proceed by analysing the actual evolution of the above one-loop RG equations. The scale evolution of the parameters of the Hamiltonian would still adequately describe the properties of the system until we reach a critical scale ξ_*, at which the strong-coupling regime begins.

The evolution of the renormalized function $g(x; \xi)$ can be analysed both numerically and analytically. It can be shown (see appendix A) that in the case $p < 4$ for a general continuous starting function $g(x; \xi = 0) \equiv g_0(x)$ the renormalized function $g(x; \xi)$ tends to zero everywhere in the interval $0 \leqslant x < (1 - \Delta(\xi))$, while in the narrow (scale-dependent) interval $\Delta(\xi)$ near $x = 1$ the values of the function $g(x; \xi)$ grow:

$$g(x; \xi) \sim \begin{cases} a\dfrac{u}{1 - u\xi} & \text{at } (1 - x) \ll \Delta(\xi) \\ 0 & \text{at } (1 - x) \gg \Delta(\xi) \end{cases} \tag{2.19}$$

$$\tilde{g}(\xi) \sim u \ln \frac{1}{1 - u\xi} \tag{2.20}$$

where

$$\Delta(\xi) \simeq (1 - u\xi)\,. \tag{2.21}$$

Here a is a positive non-universal constant, and the critical scale ξ_* is defined by the condition that the values of the renormalized parameters become of the order of one: $(1 - u\xi_*) \sim u$, or $\xi_* \sim 1/u$. Correspondingly, the spatial scale at which the system is entering the strong-coupling regime is

$$R_* \sim \exp\left(\frac{1}{u}\right). \tag{2.22}$$

Note that the value of this scale is much bigger than the usual crossover scale $\sim u^{-\alpha/\nu}$ (where α and ν are the pure system specific heat and the correlation length critical exponents), at which the disorder becomes relevant for the critical behaviour.

According to the above result, the value of the narrow band near $x = 1$ where the function $g(x; \xi)$ formally becomes divergent is $\Delta(\xi) \simeq (1 - u\xi) \to u \ll 1$ as $\xi \to \xi_*$.

Besides, it can also be shown (appendix A) that the value of the integral $\overline{g}(\xi) \equiv \int_0^1 g(x;\xi)$ formally becomes divergent logarithmically as $\xi \to \xi_*$:

$$\overline{g}(\xi) \sim u \ln \frac{1}{1-u\xi}. \tag{2.23}$$

Qualitatively similar asymptotic behaviour for $g(x;\xi)$ is obtained for the case when the starting function $g_0(x)$ has the one-step RSB structure (2.12), and the coordinate of the step x_0 is in the 'instability region' (or for any x_0 in the Ising case $p = 1$):

$$g(x;\xi) \sim \begin{cases} \dfrac{g_1(0)}{1-(4-2p+px_0)g_1(0)\xi} & \text{at } x_0 < x < 1 \\ 0 & \text{at } 0 \leqslant x < x_0 . \end{cases} \tag{2.24}$$

Here $g_1(0) \equiv g_1(\xi = 0) \sim u$, and the coefficient $(4 - 2p + px_0)$ is always positive. In this case again, the system arrives at the strong-coupling regime at scales $\xi \sim 1/u$.

Note that the above asymptotics do not explicitly involve ϵ. Actually, the role of the parameter $\epsilon > 0$ is to 'push' the RG trajectories out of the trivial Gaussian fixed point $g = 0$; $\tilde{g} = 0$. Thus, the value of ϵ, as well as the values of the starting parameters $g_0(x)$, $\tilde{g}_0$, define a scale at which the solutions finally arrive at the above asymptotic regime. In the case $\epsilon < 0$ (above four dimensions) the Gaussian fixed point is stable; on the other hand, the strong-coupling asymptotics still exists in this case as well, separated from the trivial asymptotics by a finite (depending on the value of ϵ) barrier. Therefore, although *infinitely small* disorder remains irrelevant for the critical behaviour above four dimensions, if the disorder is strong enough (bigger than a certain value dependent on the ϵ threshold) the RG trajectories could arrive again at the above strong-coupling regime.

3. Scaling and correlation functions

3.1. Temperature scales

The renormalization of the mass term $\tau(\xi)\sum_{a=1}^{n}\phi_a^2$ is described by the following RG equation:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\ln\tau = 2 - \frac{1}{8\pi^2}\left[(2+p)\tilde{g} + p\sum_{a\neq 1}^{n} g_{1a}\right]. \tag{3.1}$$

Changing (as in the previous section) $g_{ab} \to 8\pi^2 g_{ab}$, and $g_{a\neq b} \to -g_{a\neq b}$, in the Parisi representation we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\ln\tau = 2 - \left[(2+p)\tilde{g}(\xi) + p\int_0^1 g(x;\xi)\right] \tag{3.2}$$

or

$$\tau(\xi) = \tau_0 \exp\left\{2\xi - \int_0^{\xi} \mathrm{d}\eta[(2+p)\tilde{g}(\eta) + p\overline{g}(\eta)]\right\} \tag{3.3}$$

where $\tilde{g}(\eta)$ and $\overline{g}(\eta) \equiv \int_0^1 \mathrm{d}x\, g(x;\eta)$ are the solutions of the RG equations (2.10), (2.11).

Consider first what is going on in the traditional RS situation. The RS interaction parameters $\tilde{g}(\xi)$ and $g(\xi)$ arrive at the fixed point $(\tilde{g}_{\mathrm{rs}}, g_{\mathrm{rs}})$ (2.14), and then for the dependence of the renormalized mass $\tau(\xi)$, according to (3.3), one gets

$$\tau(\xi) = \tau_0 \exp\{\Delta_\tau \xi\} \tag{3.4}$$

where

$$\Delta_\tau = 2 - [(2+p)\tilde{g}_{\rm rs} + p g_{\rm rs}]\,. \tag{3.5}$$

At scale ξ_c, such that $\tau(\xi_c) \sim 1$, the system gets out of the scaling region. Since the RG parameter $\xi \equiv \ln R$, where R is the spatial scale, according to (3.4) one finds the correlation length R_c as a function of the reduced temperature τ_0

$$R_c(\tau_0) \sim \tau_0^{-\nu} \tag{3.6}$$

where $\nu = 1/\Delta_\tau$.

Actually, if the starting value of the disorder parameter $g(\xi = 0) \equiv u$ is much smaller than the value of the pure system interaction $\tilde{g}(\xi = 0) \equiv g_0$, the situation is somewhat more complicated. In this case the RG flow for $\tilde{g}(\xi)$ first arrives at the pure system fixed point $\tilde{g}_{\rm (pure)}$, as if the disorder perturbation does not exist. Then, since the pure system fixed point is unstable with respect to the disorder perturbations, at scales bigger than a certain disorder dependent scale ξ_u, the RG trajectories eventually arrive at the disorder-induced fixed point $(\tilde{g}_{\rm rs}, g_{\rm rs})$. According to the traditional theory [4], $\xi_u \sim \frac{\nu}{\alpha}\ln\frac{1}{u}$, and the corresponding spatial scale is $R_u \sim u^{-\nu/\alpha} \gg 1$.

Coming back to the scaling behaviour of the mass parameter $\tau(\xi)$, equation (3.4), we see that if the value of the temperature τ_0 is such that $\tau(\xi)$ becomes of the order of one before the crossover scale ξ_u is reached, then for the scaling behaviour of the correlation length one finds essentially the pure system result $R_c(\tau_0) \sim \tau_0^{-\nu_{\rm (pure)}}$. Thus, the pure system critical behaviour is observed only for temperatures not too close to T_c: $\tau_u \ll \tau_0 \ll 1$.

On the other hand, if $\tau_0 \ll \tau_u$, the RG trajectories for $\tilde{g}(\xi)$ and $g(\xi)$ finally arrive (after crossover) at the new universal disorder-induced fixed point $(\tilde{g}_{\rm rs}, g_{\rm rs})$. In this case scaling behaviour of the correlation length, becomes controlled by the new universal critical exponent ν defined by the fixed point $(\tilde{g}_{\rm rs}, g_{\rm rs})$ of the random system.

Consider now what happens if the RSB RG scenario takes place. For the same reasons as discussed above, if the disorder parameter u is small, the critical behaviour in the temperature interval $\tau_u \ll \tau_0 \ll 1$ is essentially controlled by the pure system fixed point, and the presence of disorder is irrelevant.

However, at temperatures $\tau_0 \ll \tau_u$ the situation becomes completely different from the RS case. According to the solutions (2.19), (2.24), at scales $\xi_u \ll \xi \ll \xi_* \sim \frac{1}{u}$ the parameters $\tilde{g}(\xi)$ and $g(x;\xi)$ do not arrive at any fixed point. Therefore, according to equation (3.3), the correlation length becomes defined by the following non-trivial equation:

$$2\ln R_c - \int_0^{\ln R_c} {\rm d}\eta[(2+p)\tilde{g}(\eta) + p\overline{g}(\eta)] = \ln\frac{1}{\tau_0}\,. \tag{3.7}$$

Thus, as the temperature becomes sufficiently close to T_c (in the disorder-dominated region $\tau_0 \ll \tau_u$) there will be no usual scaling behaviour of the correlation length.

Finally, as the temperature parameter τ_0 becomes smaller and smaller, at the scale $\xi_* \equiv \ln R_* \sim \frac{1}{u}$ the system enters the strong-coupling regime. Here the interaction parameters $\tilde{g}(\xi)$ and $g(x;\xi)$ become non-small, while the renormalized mass $\tau(\xi)$ still remains small. According to the solution obtained in appendix A, the integrals $\int_0^{\xi_*}{\rm d}\eta\,\tilde{g}(\eta)$ and $\int_0^{\xi_*}{\rm d}\eta\,\overline{g}(\eta)$ have finite (depending on the initial conditions) values. Thus, according to equation (3.7), for the crossover temperature we get

$$\tau_* \sim \exp\left(-\frac{\rm constant}{u}\right). \tag{3.8}$$

In the close vicinity of T_c, at $\tau \ll \tau_*$, we face the situation that at large scales the interaction parameters of the asymptotic (zero-mass) Hamiltonian become non-small, and the properties of the system cannot be analysed in terms of a simple one-loop RG approach. Note, however, that it is the parameter describing the disorder, $g(x;\xi)$, which is the most divergent. Therefore, the qualitative structure of the asymptotic Hamiltonian makes it possible to speculate that in the temperature interval $\tau \ll \tau_*$ near T_c the critical properties of the system should be essentially SG-like (see discussion in section 5).

3.2. Specific heat

According to the standard procedure the leading singularity of the specific heat can be calculated as follows:

$$C \sim \int d^D R[\overline{\langle\phi^2(0)\phi^2(R)\rangle} - \overline{\langle\phi^2(0)\rangle\langle\phi^2(R)\rangle}]. \tag{3.9}$$

In terms of the RG scheme for the correlation function

$$W(R) \equiv \overline{\langle\phi^2(0)\phi^2(R)\rangle} - \overline{\langle\phi^2(0)\rangle\langle\phi^2(R)\rangle} \tag{3.10}$$

one gets

$$W(R) = (G_0(R))^2 m^2(R) \tag{3.11}$$

where $G_0(R) = R^{-(D-2)}$ is the free field two-point correlation function, and the mass-like object $m(R)$ is given by the solution of the following (one-loop) RG equation:

$$\frac{d}{d\xi}\ln m(\xi) = -\left[(2+p)\tilde{g}(\xi) - p\sum_{a\neq 1}^{n} g_{a1}(\xi)\right]. \tag{3.12}$$

Here, as usual, $\xi = \ln R$, and the renormalized interaction parameters $\tilde{g}(\xi)$ and $g_{a\neq b}(\xi)$ are the solutions of the replica RG equations (2.5), (2.6). In the Parisi representation, $g_{a\neq b}(\xi) \to g(x;\xi)$, one gets

$$m(R) = \exp\left\{-(2+p)\int_0^{\ln R} d\xi\,\tilde{g}(\xi) - p\int_0^{\ln R} d\xi \int_0^1 dx\, g(x;\xi)\right\}. \tag{3.13}$$

Then, after simple transformations for the singular part of the specific heat, equation (3.9), one obtains:

$$C \sim \int_0^{\xi_{\max}} d\xi \exp\left\{\epsilon\xi - 2(2+p)\int_0^{\xi} d\eta\,\tilde{g}(\eta) - 2p\int_0^{\xi} d\eta\,\overline{g}(\eta)\right\} \tag{3.14}$$

where $\overline{g}(\eta) \equiv \int_0^1 dx\, g(x;\eta)$. The infrared cut-off $\xi_{\max}$ in (3.14) is the scale at which the system gets out of the scaling regime, and if the traditional scaling situation takes place, one finds that $\xi_{\max} \sim \ln(1/\tau_0)$.

One can easily check that using equation (3.14) for the RS and one-step RSB fixed points, equations (2.14) and (2.15), the usual scaling for the specific heat $C(\tau) \sim \tau^{-\alpha}$ can be recovered, and known results (in particular, equation (2.18)) for the corresponding critical exponents can be obtained.

However, if the considered problem is characterized by the RSB of a general type the situation becomes completely different. According to the RG strong-coupling asymptotic solution (2.20), (2.23), in the disorder-dominated region $\tau_* \ll \tau_0 \ll u^{\nu/\alpha}$ (which corresponds to scales $\xi_u \ll \xi \ll \xi_*$) the RG trajectories for the interaction parameters $\tilde{g}(\xi)$ and $\overline{g}(\xi)$ do not arrive at any fixed point, and according to equation (3.14) one finds that the specific heat becomes a complicated function of the temperature parameter τ_0. Thus, in this case in the disorder-dominated critical region the temperature dependence of the specific heat does not have the traditional scaling form.

Moreover, in the 'SG-like' temperature interval in the close vicinity of T_c, where the renormalized interaction parameters $\tilde{g}$ and $\overline{g}$ become non-small, one finds that the integrals over ξ in equation (3.14) become convergent. It means that the temperature-dependent upper cut-off scale $\xi_{\max}$ becomes irrelevant. Thus, one finds that the 'would be singular part' of the specific heat is 'smoothed out' in the temperature interval $\sim\tau_*$ around T_c. In other words, the specific heat becomes *non-singular* at the phase transition point.

3.3. Correlation functions

As noted in section 2, the calculations of the critical behaviour of the usual two-point correlation function involve a next-order (two-loop) approximation of the RG analysis. In the framework of the present RSB scheme such types of calculations appear to be extremely cumbersome, and have not yet been done. On the other hand, the scaling properties of the so-called spin-glass-type connected correlation function

$$K(R) = \overline{(\langle\phi(0)\phi(R)\rangle - \langle\phi(0)\rangle\langle\phi(R)\rangle)^2} \equiv \overline{\langle\langle\phi(0)\phi(R)\rangle\rangle^2} \tag{3.15}$$

can be calculated within the usual one-loop approximation, and it is the properties of such a type of correlation function that are of main interest for the considered spin-glass effects in the critical phenomena (see section 5).

It is well known [6] that in terms of the replica formalism the correlation function (3.15) can be represented as follows:

$$K(R) = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b}^{n} K_{ab}(R) \tag{3.16}$$

where

$$K_{ab}(R) = \langle\langle\phi_a(0)\phi_b(0)\phi_a(R)\phi_b(R)\rangle\rangle\,. \tag{3.17}$$

It is also well known that in terms of the standard RG formalism for the scaling behaviour of the above replica correlation function one finds

$$K_{ab}(R) \sim (G_0(R))^2 (Z_{ab}(R))^2 \tag{3.18}$$

where

$$G_0(R) = R^{-(D-2)} \tag{3.19}$$

is the free-field correlation function, and in the one-loop approximation the scaling of the mass-like object $Z_{ab}(R)$ (with $a \neq b$) is defined by the following RG equation:

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \ln Z_{ab}(\xi) = 2g_{ab}(\xi)\,. \tag{3.20}$$

Here $g_{a\neq b}(\xi) > 0$ is the solution of the corresponding RG equations (2.5), (2.6), $\xi = \ln R$, and $Z_{ab}(0) \equiv 1$.

Thus, for the correlation function (3.18) one finds

$$K_{ab}(R) \sim (G_0(R))^2 \exp\left\{4\int_0^{\ln R} \mathrm{d}\xi\, g_{ab}(\xi)\right\}. \tag{3.21}$$

Correspondingly, in the Parisi representation: $g_{a\neq b}(\xi) \to g(x;\xi)$ and $K_{a\neq b}(R) \to K(x;R)$, one gets

$$K(x;R) \sim (G_0(R))^2 \exp\left\{4\int_0^{\ln R} \mathrm{d}\xi\, g(x;\xi)\right\}. \tag{3.22}$$

Consider separately the results for the above correlation function given by the traditional RS fixed point, the one-step RSB fixed point and the strong-coupling asymptotics.

3.3.1. Replica-symmetric fixed point. In the traditional RS case the interaction parameter $g_{a\neq b}(\xi) \equiv u(\xi)$ arrives at the fixed point

$$u_* = \epsilon \frac{4-p}{16(p-1)} \qquad (1 < p < 4)$$

and according to equations (3.21), (3.16) one obtains simple scaling

$$K_{\mathrm{rs}}(R) \sim R^{-2(D-2)+\theta_{\mathrm{RS}}} \tag{3.23}$$

with the universal disorder-induced critical exponent

$$\theta_{\mathrm{RS}} = \epsilon \frac{4-p}{4(p-1)} . \tag{3.24}$$

3.3.2. One-step RSB fixed point In the case of the one-step RSB fixed point, equation (2.15), the situation becomes somewhat more complicated. Here one finds that the correlation function $K(x; R)$ also has one-step RSB structure:

$$K(x; R) \sim \begin{cases} K_0(R) & \text{for } 0 \leqslant x < x_0 \\ K_1(R) & \text{for } x_0 < x \leqslant 1 \end{cases} \tag{3.25}$$

where

$$K_0(R) \sim R^{-2(D-2)} = G_0^2(R) \qquad K_1(R) \sim R^{-2(D-2)+\theta_{\mathrm{1RSB}}} \tag{3.26}$$

with the *non-universal* critical exponent θ_{1RSB} explicitly depending on the coordinate of the step x_0:

$$\theta_{\mathrm{1RSB}} = \epsilon \frac{4(4-p)}{16(p-1) - p x_0 (8+p)} \qquad (> \theta_{\mathrm{RS}}) . \tag{3.27}$$

Since the critical exponent θ_{1RSB} is positive, the leading contribution to the asymptotic behaviour of the 'observable' quantity $K(R) = \overline{\langle\langle\phi(0)\phi(R)\rangle\rangle^2}$, equation (3.16), is defined only by the function $K_1(R)$

$$K(R) = \int_0^1 \mathrm{d}x\, K(x; R) \sim (1-x_0)K_1(R) + x_0 K_0(R) \sim R^{-2(D-2)+\theta_{\mathrm{1RSB}}} . \tag{3.28}$$

Thus, in this case the correlation function $K(R)$ decays more slowly than in the RS case (3.23), (3.24).

3.3.3. Strong coupling asymptotics. In the case of a general type of RSB, according to the qualitative solution (2.19), (2.20), the function $g(x; \xi)$ does not arrive at any fixed point at scales $\xi \gg \xi_u \sim \frac{\nu}{\alpha} \ln \frac{1}{u}$. Therefore, at the disorder-dominated scales $R \gg R_u \sim u^{-\nu/\alpha} \gg 1$ there must be no scaling behaviour of the correlation function $K(R)$. Near the critical scale $\xi_* \sim 1/u$ the qualitative behaviour of the solution $g(x; \xi)$ is given by equation (2.19). Therefore, according to equation (3.22), near the critical scale $R_* \sim \exp(1/u)$ for the correlation function $K(x; R)$ one obtains

$$K(x; R) \sim \begin{cases} R^{-2(D-2)}(1 - u \ln R)^{-4a} \equiv K_1(R) & \text{for } (1-x) \ll \Delta(R) \\ R^{-2(D-2)} = G_0^2(R) \equiv K_0(R) & \text{for } (1-x) \gg \Delta(R) \end{cases} \tag{3.29}$$

where $\Delta(R) = (1 - u \ln R) \to u \ll 1$ as $R \to R_*$.

At the critical scale, where $(1 - u \ln R_*) \sim u$, according to equation (3.29) the shape of the replica function $K(x; R)$ becomes 'quasi-one-step':

$$K(x; R_*) \sim \begin{cases} u^{-4a} \exp\left\{-\dfrac{2(D-2)}{u}\right\} \equiv K_1^* & \text{for } (1-x) \ll u \\ \exp\left\{-\dfrac{2(D-2)}{u}\right\} \equiv K_0^* & \text{for } (1-x) \gg u\,. \end{cases} \tag{3.30}$$

Note that although both values K_1^* and K_0^* are exponentially small, their ratio $K_1^*/K_0^* \sim u^{-4a}$ is big.

Finally, at scales $R \gg R_*$ the system enters the strong-coupling regime, where the simple one-loop RG approach cannot be used any more.

Physical interpretation of the above results will be discussed in section 5.

4. Marginal case $p = 4$

In systems with the number of spin components $p = 4$ (in which the pure system specific heat critical exponent $\alpha = 0$) the disorder appears to be marginally irrelevant in a sense that it does not change the critical exponents. Nevertheless, the critical behaviour (described in terms of the logarithmic singularities) is affected by the disorder, and moreover, the RSB phenomena appear to be relevant in this case as well.

Consider first the RS situation: $g(x; \xi) \equiv g(\xi)$. For the RG equations (2.10), (2.11) one gets

$$\frac{dg}{d\xi} = (\epsilon - 12\tilde{g})g - 4g^2 \qquad \frac{d\tilde{g}}{d\xi} = (\epsilon - 12\tilde{g})\tilde{g} + 4g^2\,. \tag{4.1}$$

In the pure system ($g \equiv 0$) the fixed point is

$$\tilde{g}_{\text{pure}} = \tfrac{1}{12}\epsilon\,. \tag{4.2}$$

Using equation (3.14) for the singular part of the specific heat of the pure system one easily finds

$$C_{\text{pure}}(\tau) \sim \ln\left(\frac{1}{\tau}\right). \tag{4.3}$$

Thus, although the specific-heat critical exponent of the pure system is zero, the specific heat is still divergent in the critical point.

For the system with disorder, the (RS) asymptotic solution of equations (4.1) is

$$g(\xi) \simeq \tfrac{1}{4}\xi^{-1} \to 0 \qquad \tilde{g}(\xi) \simeq \tfrac{1}{12}\epsilon + q(\xi) \tag{4.4}$$

where

$$q(\xi) \sim \xi^{-2} \to 0\,. \tag{4.5}$$

In this case the renormalized parameters asymptotically approach the pure-system fixed point $\tilde{g} = \epsilon/12$, $g = 0$ (so that the disorder is marginally irrelevant). Nevertheless, due to slow power-law approach to the fixed point the logarithmic singularity of the specific heat changes into another universal type. From the general expression (3.14) for the singular part of the specific heat one obtains

$$C \sim \int_0^{\ln(1/\tau)} d\xi \exp\left\{\int_0^{\xi} d\eta [\epsilon - 12\tilde{g}(\eta) - 8g(\eta)]\right\}. \tag{4.6}$$

Using the result (4.4) one easily finds

$$C_{\mathrm{rs}}(\tau) \sim \frac{1}{\ln(1/\tau)} . \tag{4.7}$$

One can also easily check that (unlike the systems with $p < 4$) the crossover from pure system critical behaviour, equation (4.3), to disorder induced, equation (4.7), takes place in the exponentially small temperature interval near T_c:

$$\tau_u \sim \exp\left(-\frac{1}{u}\right). \tag{4.8}$$

Consider now the effects of the RSB. The analytic solution of the RG equations (2.10), (2.11) (see appendix B) shows that there is no strong-coupling regime in the $p = 4$ case, and the asymptotic behaviour (at scales $\xi \gg 1/u$) of the renormalized parameters can be found exactly:

$$g(x;\xi) \sim \begin{cases} \xi^{-2} & (1-x) \gg \dfrac{1}{\sqrt{\gamma\xi}} \\ \dfrac{1}{\sqrt{\gamma\xi}} & (1-x) \ll \dfrac{1}{\sqrt{\gamma\xi}} \end{cases} \tag{4.9}$$

$$\tilde{g}(\xi) \simeq \frac{\epsilon}{12} + q(\xi) \qquad q(\xi) \sim \xi^{-3/2} \to 0 . \tag{4.10}$$

Here $\gamma \equiv g_0'(x = 1) \sim u$ is the derivative of the starting RSB function $g_0(x)$ at $x = 1$.

As in the RS case the renormalized parameters asymptotically approach the pure system fixed point $\tilde{g} = \epsilon/12$, $g(x) = 0$. Nevertheless, the structure of the asymptotic solution for the renormalized function $g(x;\xi)$ near this fixed point exhibits strong RSB.

However, the specific heat appears not to be affected by the RSB. According to equation (3.14) the leading singularity of the specific heat is defined by the integral $\int_0^1 \mathrm{d}x\, g(x;\xi) \equiv \overline{g}(\xi)$ and not by the function $g(x;\xi)$ itself. It can be shown (see equation (B.12)) that in the asymptotic regime the value of $\overline{g}(\xi)$ coincides with the RS asymptotics (4.4):

$$\overline{g}(\xi) \sim \tfrac{1}{4}\xi^{-1} . \tag{4.11}$$

Therefore, for the specific heat singularity one obtains the result coinciding with the RS one, equation (4.7).

On the other hand, the asymptotic behaviour of the correlation functions appears to be quite different from the results of the traditional RS solution. In the RS case, equation (4.4), according to equation (3.22) for the correlation function

$$K(R) = \overline{\langle\langle\phi(0)\phi(R)\rangle\rangle^2} \tag{4.12}$$

one easily finds the following result:

$$K(R) \sim (G_0(R))^2 \exp\left\{4\int_0^{\ln R} \mathrm{d}\xi\, g(\xi)\right\} = (G_0(R))^2 \ln R . \tag{4.13}$$

Therefore, in the RS case the disorder provides only the logarithmic correction to the correlation function.

In the case of the RSB solution, equation (4.9), according to equation (3.22) for the replica correlation function $K(x;R)$, one easily finds

$$K(x;R) \sim \begin{cases} (G_0(R))^2 \exp\{(\text{constant})\sqrt{\gamma \ln R}\} & (1-x)\sqrt{\gamma \ln R} \gg 1 \\ (G_0(R))^2 & (1-x)\sqrt{\gamma \ln R} \ll 1 . \end{cases} \tag{4.14}$$

Correspondingly, for the 'observable' correlation function, equation (4.12), one eventually obtains

$$K(R) = \int_0^1 \mathrm{d}x\, K(x; R) \sim (G_0(R))^2 \exp\{(\text{constant})\sqrt{\gamma \ln R}\}. \tag{4.15}$$

This result is essentially different from the RS one, equation (4.13).

5. Discussion

According to the results obtained in this paper, as well as in [1], spontaneous replica symmetry breaking in the effective interaction potential for the fluctuating fields has a dramatic effect on the renormalization group flows and on the critical properties. In the systems with the number of spin components $p < 4$ the traditional replica-symmetric RG flows at dimensions $D = 4 - \epsilon$, which are usually considered as describing the disorder-induced universal critical behaviour, appear to be unstable with respect to 'turning on' the RSB potentials. Moreover, for a general type of the Parisi RSB structures there exist no stable fixed points, and the RG flows lead to the *strong-coupling regime* at the finite scale $R_* \sim \exp(1/u)$, where u is the small parameter describing the disorder. Unlike the systems with $1 < p < 4$, where there exist stable fixed points having one-step RSB structures, equation (2.15), in the Ising case, $p = 1$, there exist no stable fixed points, and any RSB interactions lead to the strong-coupling regime.

The problem now is to understand how all those formal results should be interpreted in qualitative terms for the observable physics. The first qualitative result which comes out from the calculations of section 3 is that the physical quantities exhibit no scaling behaviour in the critical region. Actually, this is just the general consequence of the absence of the fixed point in the RG. Although it is the RSB phenomena which provide the absence of the fixed point in the considered problem, the non-scaling behaviour itself does not give insight into specific effects of the RSB which is the main interest of the present study.

The key question, which remains unanswered, is whether or not the obtained RSB strong-coupling phenomena in the RG flows could be interpreted as the onset of a kind of spin-glass phase near T_c. Since it is the RSB interaction parameter describing the disorder, $g(x; \xi)$, which is the most divergent, it is tempting to argue that in the temperature interval $\tau \ll \tau_* \sim \exp(-1/u)$ near T_c the properties of the system should be essentially SG-like.

It should be stressed, however, that in the present study we observe only the crossover temperature τ_*, at which the change of the critical regime may occur, and it is hardly possible to associate this temperature with any kind of phase transition. Actually, if the RSB effects could indeed provide any kind of true thermodynamic order parameter, then this must be true in the whole temperature interval where the RSB potentials exist.

The true spin-glass order (in the traditional sense) arises from the onset of the non-zero order parameter $Q_{ab}(x) = \langle\phi_a(x)\phi_b(x)\rangle; a \neq b$, and, at least for the infinite-range spin glasses, Q_{ab} develops hierarchical dependence on replica indices obtained by Parisi [16]. In the present problem we find only that the coupling matrix g_{ab} for the fluctuating fields develops strong RSB structure and its elements become non-small at a finite scale. Therefore, it seems more realistic to interpret discovered RSB strong-coupling phenomena in the RG just as a new type of critical behaviour characterized by strong SG effects in the scaling properties rather then in the ground state.

In spin glasses it is generally believed that RSB phenomena can be interpreted as a factorization of the phase space into an (ultrametric) hierarchy of 'valleys', or local minima pure states, separated by macroscopic (infinite) barriers [6]. Although in the systems considered here the local minima configurations responsible for the RSB are not likely to be separated by infinite barriers (otherwise it would mean true SG freezing), it would be natural to interpret obtained phenomena as effective factorization of the phase space into a hierarchy of valleys separated by *finite* barriers. In this situation one could expect that besides the usual critical slowing down (corresponding to the relaxation inside one valley), qualitatively *much bigger (exponentially large) relaxation times would be required for overcoming* barriers separating different valleys. Therefore, the traditional measurements (made at finite equilibration times) can actually correspond to the equilibration within one valley only, and not to the true thermal equilibrium. Then in a close vicinity of the critical point different measurements of the critical properties of, e.g., spatial correlation functions (in the same sample) would exhibit different results as if the state of the system becomes effectively 'trapped' in different valleys, and thus the traditional spin-glass situation will be observed.

In this respect the *replica structure* of the two-point correlation function $K_{ab}(R) = \langle\phi_a(0)\phi_b(0)\phi_a(R)\phi_b(R)\rangle$ (studied in section 3.3) is of fundamental importance. Of course, the 'observable' quantity (averaged over the disorder)

$$K(R) \equiv \overline{\langle\phi(0)\phi(R)\rangle^2} = \lim_{n\to 0}\frac{1}{n(n-1)}\sum_{a\neq b}^{n} K_{ab}(R) \tag{5.1}$$

has no replica structure, and correspondingly, besides its scaling properties, it cannot give any direct information about the RSB. Nevertheless, the qualitative difference between the traditional RS critical phenomena and those with RSB is not only in the fact that in the case of the one-step RSB fixed point the corresponding critical exponent θ, equation (3.27), of the correlation function $K(R)$ is non-universal, or in the case of the strong-coupling regime the function $K(R)$ exhibits no scaling.

According to the traditional SG philosophy [6, 14] the result that the behaviour of the replica quantity $K_{ab}(R)$ depends on the replica indices (a, b) indicates that in different measurements of the correlation functions for a *a given realization* of the disorder one has to obtain *different* results. To describe this phenomenon in more concrete terms, such as in spin glasses, it appears to be more convenient to deal with the so-called 'overlap' quantities. In our case the *spatially averaged* overlap quantity, which corresponds to the *replica* correlation function $K_{ab}(R)$, could be defined as follows:

$$K_{ij}(R) \equiv \frac{1}{V}\int \mathrm{d}^D r\langle\phi(r)\phi(r+R)\rangle_i\langle\phi(r)\phi(r+R)\rangle_j\,. \tag{5.2}$$

Here i and j label two different realizations of the disorder, and it is assumed (as in spin glasses) that the measurable thermal average corresponds to a particular valley in the space of states for a given realization of the disorder, and not to the true thermal average.

Apparently, if the usual RS situation takes place (so that only one global valley exists), then for the correlation function $K_{ij}(R)$ one will obtain the same result $K(R)$, equation (5.1), in all measurements for any pair of realizations of the disorder. However, in the case of the RSB solution, the situation becomes qualitatively different. The point is that if the phase space is factorized into a multiple valley structure (and each thermal average corresponds to one valley only), then the correlation function (5.2), although being spatially averaged, becomes a *probabilistic* quantity, and according to the SG theory of the RSB, the fluctuations of $K_{ij}(R)$ are described by the distribution function, which is defined by the RSB structure of the replica quantity $K_{ab}(R)$.

In particular, if the one-step RSB result (3.25) takes place, then for the correlation function $K_{ij}(R)$ one has to obtain the values $K_0(R)$ and $K_1(R)$, with the probabilities x_0 and $(1 - x_0)$ correspondingly. Similarly, in the case of the non-scaling result (3.29), which also exhibits quasi-one-step RSB asymptotic structure, the probabilities of obtaining the values $K_0(R)$ and $K_1(R)$ must be $(1 - \Delta(R))$ and $\Delta(R)$ correspondingly.

The other unsolved problem is about the possible relevance of the RSB phenomena considered here and in [1] for the so-called Griffith phase [13].

Apparently, if the summation over multiple local minima configurations yields the effective Hamiltonian with the RSB in the fourth-order potential, then formally one is dealing with a phase exhibiting a different symmetry than the conventional RS paramagnetic phase. Thus there would have to be a temperature T_{RSB} at which this change in symmetry occurs, since for large enough τ there must be no RSB. Although in the present study we were concentrated on the effects of the RSB for the fluctuating fields and for the critical phenomena, it should be stressed that it is the statistics of the saddle-point solutions only, which is responsible for the appearance of the RSB. Therefore, one can consider the separate problem of summing over saddle-point solutions without fluctuating fields, keeping arbitrary parameter τ, and aiming to find a finite value of τ_{RSB} at which the RSB solution for this problem disappears.

Of course, in general this problem is very difficult to solve, but one can easily obtain an estimate for the value of τ_{RSB} (assuming that at $\tau = 0$ the RSB situation takes place). According to the qualitative study of this problem in [1], the RSB solution can occur only when the effective interactions between the 'islands', where the system is effectively below T_c, become non-small. The islands are the regions where $\delta\tau(x) > \tau$. According to the Gaussian distribution for $\delta\tau(x)$, the average distance between them must be of the order of $\exp[-\tau^2/u]$, so that the islands become distant at $\tau > \sqrt{u}$. The interaction between the islands is exponentially small in their separation. Therefore at $\tau > \sqrt{u}$ they must become weakly interacting, and there must be no RSB.

Note now that the shift of T_c with respect to the corresponding pure system is also of the order of $\sqrt{u}$. On the other hand, it is the presence of multiple local minima configurations which is believed to be the fundamental reason for the existence of the Griffith phase [13] which is claimed to be observed in the temperature interval between T_c of the disordered system and T_c of the corresponding pure system. On these grounds it is tempting to associate the (hypothetical) RSB transition in the statistics of the saddle-point solutions with the Griffith transition. Correspondingly, it would also be natural to suggest that discovered RSB phenomena in the scaling properties of weakly disordered systems could be associated with the Griffith effects.

Acknowledgments

VD acknowledges Laboratiore de Physique Theorique de Ecole Normale Superieure for hospitality. Numerous fruitful and encouraging discussions with M Mezard and Vl S Dotsenko are also greatly acknowledged.

This work has been supported in part by the INTAS Grant No 1010-CT93-0027 and by the Grant of the Russian Fund for Fundamental Research 93-02-2081.

Appendix A. The asymptotic solution for the $p < 4$ case

In this appendix we derive the asymptotic solution of the RG equations (2.10), (2.11):

$$\frac{\mathrm{d}}{\mathrm{d}\xi} g(x) = (\epsilon - (4+2p)\tilde{g})g(x) + 4g^2(x) - 2pg(x)\int_0^1 \mathrm{d}y\, g(y) - p\int_0^x \mathrm{d}y\,(g(x)-g(y))^2 \tag{A.1}$$

$$\frac{d}{d\xi}\tilde{g} = \epsilon\tilde{g} - (8+p)\tilde{g}^2 + p\overline{g^2} \tag{A.2}$$

(where $\overline{g^2} \equiv \int_0^1 \mathrm{d}x\, g^2(x)$) for the number of components $p < 4$.

It can be shown *a posteriori* that the term $(\epsilon - (4+2p)\tilde{g})g(x)$ in equation (A.1) is irrelevant in the asymptotic regime. So, consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}g(x) = 4g^2(x) - 2pg(x)\int_0^1 \mathrm{d}y\, g(y) - p\int_0^x \mathrm{d}y\,(g(x)-g(y))^2 . \tag{A.3}$$

After taking the derivative over x and after simple transformations one gets

$$\frac{\mathrm{d}}{\mathrm{d}\xi}g'(x) = 2pg'(x)\left[(\lambda-1)g(x) - \int_x^1 \mathrm{d}y\,(1-y)g'(y)\right] \tag{A.4}$$

where $\lambda = 4/p > 1$. Let us introduce

$$V(x) \equiv \int_x^1 \mathrm{d}y\,(1-y)g'(y) . \tag{A.5}$$

According to this definition one has

$$g'(x) = -\frac{1}{1-x}V'(x)$$
$$g(x) = \int_0^x \mathrm{d}y\, g'(y) = -\int_0^x \mathrm{d}y \frac{1}{1-y}V'(y) . \tag{A.6}$$

Here for simplicity we consider the case $g(x=0) = 0$ (the behaviour of the solution for $g(x=0) \neq 0$ in the asymptotic regime can be shown to be qualitatively the same).

Then, for equation (A.4) after simple transformations we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi}V'(x) = -2pV'(x)\left[\int_0^x \mathrm{d}y\frac{\lambda-y}{1-y}V'(y) + \overline{g}(\xi)\right] \tag{A.7}$$

where $\overline{g}(\xi) \equiv \int_0^1 \mathrm{d}x\, g(x,\xi) = \int_0^1 \mathrm{d}x\,(1-x)g'(x) = V(x=0;\xi)$.

Let us now define

$$W(x;\xi) = \int_0^x \mathrm{d}y\,\frac{\lambda-y}{1-y}V'(y) \tag{A.8}$$

or

$$V'(x) = \frac{1-x}{\lambda-x}W'(x) . \tag{A.9}$$

From equation (A.7) one gets

$$\frac{\mathrm{d}}{\mathrm{d}\xi}W'(x) = -2pW'(x)[W(x) + \overline{g}(\xi)] . \tag{A.10}$$

Integrating over x yields

$$\frac{\mathrm{d}}{\mathrm{d}\xi}W(x) = -pW^2(x) - 2pW(x)\overline{g}(\xi) \tag{A.11}$$

(here the integration constant is zero because $W(x=0) \equiv 0$). This equation can be easily solved for any given function $\overline{g}(\xi)$:

$$W(x;\xi) = \frac{W_0(x)\exp[-2p\int_0^\xi \mathrm{d}\eta\,\overline{g}(\eta)]}{1 + pW_0(x)\int_0^\xi \mathrm{d}t\,\exp[-2p\int_0^t \mathrm{d}\eta\,\overline{g}(\eta)]} \tag{A.12}$$

where

$$W_0(x) \equiv W(x;\xi=0) = -\int_0^x \mathrm{d}y\,(\lambda - y)g_0'(y) \tag{A.13}$$

and $g_0(x) \equiv g(x;\xi=0)$. Coming back through the definitions (A.8) and (A.5) for the function $g(x;\xi)$ one gets

$$g(x;\xi) = \int_0^x \mathrm{d}y \frac{g_0'(y)\Theta(\xi)}{[1 - p\int_0^\xi \mathrm{d}\eta\,\Theta(\eta)\int_0^y \mathrm{d}z(\lambda - z)g_0'(z)]^2} \tag{A.14}$$

where

$$\Theta(\xi) = \exp\left[-2p\int_0^\xi \mathrm{d}\eta\,\overline{g}(\eta)\right]. \tag{A.15}$$

Integrating $\int_0^1 \mathrm{d}x\, g(x;\xi) \equiv \overline{g}(\xi)$ one gets the equation for the unknown function $\overline{g}(\xi)$

$$\overline{g}(\xi) = \int_0^1 \mathrm{d}y \frac{(1-y)g_0'(y)\Theta(\xi)}{[1 - p\int_0^\xi \mathrm{d}\eta\,\Theta(\eta)\int_0^y \mathrm{d}z\,(\lambda - z)g_0'(z)]^2}. \tag{A.16}$$

Now the problem is to find the asymptotic behaviour of $\overline{g}(\xi)$.

Let us introduce

$$G(\xi) \equiv \int_0^\xi \mathrm{d}\eta\,\overline{g}(\eta). \tag{A.17}$$

Integrating (A.16) we obtain

$$G(\xi) = \int_0^1 \mathrm{d}y \frac{(1-y)g_0'(y)A(\xi)}{[1 - pA(\xi)\int_0^y \mathrm{d}z\,(\lambda - z)g_0'(z)]} \tag{A.18}$$

where

$$A(\xi) = \int_0^\xi \mathrm{d}\eta \exp[-2pG(\eta)]. \tag{A.19}$$

Let us redefine

$$\psi(\xi) \equiv (A(\xi))^{-1} = \frac{1}{\int_0^\xi \mathrm{d}\eta \exp[-2pG(\eta)]}. \tag{A.20}$$

Then

$$G(\xi) = \int_0^1 \mathrm{d}y \frac{(1-y)g_0'(y)}{[\psi(\xi) - p\int_0^y \mathrm{d}z\,(\lambda - z)g_0'(z)]}. \tag{A.21}$$

Now, let us redefine again

$$\psi(\xi) = p\int_0^1 \mathrm{d}y\,(\lambda - y)g_0'(y) + \phi(\xi). \tag{A.22}$$

From equation (A.21) we get

$$G(\xi) = \int_0^1 \mathrm{d}y \frac{(1-y)g_0'(y)}{[p\int_y^1 \mathrm{d}z\,(\lambda - z)g_0'(z) + \phi(\xi)]}.. \tag{A.23}$$

Assuming that $\phi(\xi)$ is small, (A.23) can be estimated as follows:

$$G(\xi) = G_c + \int_0^1 \mathrm{d}y(1-y)g_0'(y)\left[\frac{1}{p\int_y^1 \mathrm{d}z\,(\lambda - z)g_0'(z) + \phi(\xi)} - \frac{1}{p\int_y^1 \mathrm{d}z\,(\lambda - z)g_0'(z)}\right] \tag{A.24}$$

or

$$G(\xi) = G_c - \phi(\xi) \int_0^1 \mathrm{d}y \frac{(1-y)g_0'(y)}{[p \int_y^1 \mathrm{d}z\,(\lambda - z)g_0'(z) + \phi(\xi)][p \int_y^1 \mathrm{d}z\,(\lambda - z)g_0'(z)]} \tag{A.25}$$

where

$$G_c \equiv \int_0^1 \mathrm{d}y \frac{(1-y)g_0'(y)}{p \int_y^1 \mathrm{d}z\,(\lambda - z)g_0'(z)} . \tag{A.26}$$

For $\phi(\xi) \ll 1$ the leading contribution in the integral in (A.25) comes from the vicinity of $y = 1$. Assuming that $g_0'(y = 1) = \gamma \neq 0$, this contribution can be estimated as follows:

$$\begin{aligned} G(\xi) &\simeq G_c - \phi(\xi) \int_{\dots}^1 \mathrm{d}y \frac{(1-y)\gamma}{[p\gamma(\lambda - 1)(1-y) + \phi(\xi)]p\gamma(\lambda-1)(1-y)} \\ &\simeq G_c - \frac{\phi(\xi)}{\gamma p^2(\lambda-1)^2} \ln \frac{1}{\phi(\xi)} \end{aligned} \tag{A.27}$$

such that, as $\phi \to 0$, the value of $G(\xi)$ goes to finite value G_c, but near this point the behaviour of this function is non-analytic.

Now let us assume that there exists a certain scale ξ_c, such that $\phi(\xi \to \xi_c) \to 0$, and consider the behaviour near ξ_c. Coming back to the definition (A.20) we can estimate

$$\begin{aligned} \psi(\xi) &= \left[\int_0^\xi \mathrm{d}\eta\, \exp(-2pG(\eta)) \right]^{-1} \\ &= \left(\int_0^{\xi_c} \mathrm{d}\eta\, \exp(-2pG(\eta)) - \int_\xi^{\xi_c} \mathrm{d}\eta \exp(-2pG(\eta)) \right)^{-1} \\ &\simeq \left(\int_0^{\xi_c} \mathrm{d}\eta\, \exp(-2pG(\eta)) - \exp(-2pG_c)(\xi_c - \xi) \right)^{-1} \\ &\simeq \frac{1}{\int_0^{\xi_c} \mathrm{d}\eta \exp(-2pG(\eta))} + \frac{\exp(-2pG_c)}{\left[\int_0^{\xi_c} \mathrm{d}\eta \exp(-2pG(\eta))\right]^2} (\xi_c - \xi) . \end{aligned} \tag{A.28}$$

Comparing this result with (A.22), we find that

$$\phi(\xi) \simeq a(\xi_c - \xi) \tag{A.29}$$

where the parameters ξ_c and a are defined by

$$\frac{1}{\int_0^{\xi_c} \mathrm{d}\eta \exp(-2pG(\eta))} = p \int_0^1 \mathrm{d}y\,(\lambda - y)g_0'(y) \tag{A.30}$$

and

$$a = \frac{\exp(-2pG_c)}{\left[\int_0^{\xi_c} \mathrm{d}\eta \exp(-2pG(\eta))\right]^2} = \left[p \int_0^1 \mathrm{d}y(\lambda - y)g_0'(y) \right]^2 \exp(-2pG_c) . \tag{A.31}$$

Let us estimate the parameters ξ_c and a by the order of magnitude. The characteristic value of the initial function $g_0(x)$ is of the order of $u \ll 1$, which is the characteristic value of the quenched disorder. If the initial function $g_0(x)$ does not have a special anomaly near $x = 1$, then its derivative γ must also be of the order of u. Then, the above integrals can be estimated as follows:

$$G_c = \int_0^1 \mathrm{d}y \frac{(1-y)g_0'(y)}{p \int_y^1 \mathrm{d}z\,(\lambda - z)g_0'(z)} \sim 1 \tag{A.32}$$

$$\int_0^1 \mathrm{d}y\,(\lambda - y) g_0'(y) \sim u \tag{A.33}$$

$$\int_0^{\xi_c} \mathrm{d}\eta\, \exp(-2pG(\eta)) \sim \xi_c\,. \tag{A.34}$$

Thus, from (A.30) and (A.31) for the parameters ξ_c and a we find

$$\xi_c \sim \frac{1}{u} \tag{A.35}$$

$$a \sim u^2\,. \tag{A.36}$$

Now we can describe the qualitative behaviour of the asymptotic solution. According to (A.27)

$$\begin{aligned} \overline{g}(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi} G(\xi) &\simeq \frac{a}{\gamma p^2 (\lambda - 1)^2} \ln \frac{1}{(\xi_c - \xi)} \\ &\sim u \ln \frac{1}{1 - u\xi}\,. \end{aligned} \tag{A.37}$$

Therefore the value of the integral $\int_0^1 \mathrm{d}x\, g(x;\xi) \equiv \overline{g}(\xi)$ formally becomes divergent at finite scale $\xi_c \sim 1/u$.

Coming back to the result (A.14) for the function $g(x;\xi)$ we have

$$\begin{aligned} g(x;\xi) &= \frac{\Theta(\xi)}{(\int_0^\xi \mathrm{d}\eta \Theta(\eta))^2} \int_0^x \mathrm{d}y \frac{g_0'(y)}{\left[p \int_y^1 \mathrm{d}z (\lambda - z) g_0'(z) + \phi(\xi)\right]^2} \\ &= -\left[\frac{\mathrm{d}}{\mathrm{d}\xi} \frac{1}{\int_0^\xi \mathrm{d}\eta \Theta(\eta)}\right] \int_0^x \mathrm{d}y \frac{g_0'(y)}{\left[p \int_y^1 \mathrm{d}z (\lambda - z) g_0'(z) + \phi(\xi)\right]^2} \\ &= -\left[\frac{\mathrm{d}}{\mathrm{d}\xi} \psi(\xi)\right] \int_0^x \mathrm{d}y \frac{g_0'(y)}{\left[p \int_y^1 \mathrm{d}z (\lambda - z) g_0'(z) + \phi(\xi)\right]^2} \\ &\simeq a \int_0^x \mathrm{d}y \frac{g_0'(y)}{\left[p \int_y^1 \mathrm{d}z (\lambda - z) g_0'(z) + a(\xi_c - \xi)\right]^2}\,. \end{aligned} \tag{A.38}$$

Therefore, when approaching the critical scale, $\xi \to \xi_c$, the values of $g(x;\xi)$ formally become big in the narrow interval $(1 - x) \ll \Delta(\xi)$, where

$$\Delta(\xi) \sim \frac{a}{\gamma}(\xi_c - \xi) \sim (1 - u\xi)\,. \tag{A.39}$$

In this interval:

$$\begin{aligned} g(x;\xi) &\simeq g(x = 1;\xi) \equiv g_1(\xi) \\ &\simeq a \int_{\dots}^1 \mathrm{d}y \frac{\gamma}{[p(\lambda - 1)\gamma(1 - y) + a(\xi_c - \xi)]^2} \\ &\simeq \frac{1}{p(\lambda - 1)} \frac{1}{\xi_c - \xi} \\ &\sim a \frac{u}{1 - u\xi} \end{aligned} \tag{A.40}$$

where $a \sim 1$ is a (non-universal) constant.

Therefore, the considered RG approach can be applied only up to the scales such that $(1 - u\xi) \sim u$ (until the value of the parameter g_1 becomes non-small).

Now let us come back to the equation for the diagonal parameter $\tilde{g}$ (A.2). According to the asymptotics obtained above we can estimate the value of $\overline{g^2}$. Since the leading contribution comes from the region $\Delta(\xi)$ near $x = 1$, we get

$$\overline{g^2} = \int_0^1 \mathrm{d}x\, g^2(x;\xi) \sim (1-u\xi)\frac{a^2u^2}{(1-u\xi)^2} = a^2\frac{u^2}{1-u\xi}\,. \tag{A.41}$$

Therefore, from equation (A.2) we see that $\tilde{g}$ diverges as the logarithm

$$\tilde{g}(\xi) \sim u \ln \frac{1}{1-u\xi}\,. \tag{A.42}$$

Thus, *in the region near $x = 1$ where the value of $g(x;\xi)$ was obtained to be of the order* of $u/(1-u\xi)$ the first term $(\epsilon - (4+2p)\tilde{g})g(x)$ in equation (A.1) is much smaller than the other terms:

$$\tilde{g}g(x) \sim \frac{u^2}{1-u\xi}\ln\frac{1}{1-u\xi} \ll \frac{u^2}{(1-u\xi)^2} \sim g^2(x)\,. \tag{A.43}$$

Of course, the above asymptotic solution does not make it possible to obtain the behaviour of the function $g(x;\xi)$ in the whole interval $[0,1]$ for all ξ. Nevertheless, the numerical solution of the general RG equations (A.1), (A.2) clearly demonstrates that at large scales the function $g(x;\xi)$ quickly goes to zero for all x not too close to 1, while in the narrow region near $x = 1$ the values of this function become divergent. Thus, the behaviour of the asymptotic solution for $g(x;\xi)$ in the vicinity of the critical scale ξ_c could be qualitatively represented as follows:

$$g(x;\xi) \sim \begin{cases} a\dfrac{u}{1-u\xi} & \text{for } (1-x) \ll \Delta(\xi) \\ 0 & \text{for } (1-x) \gg \Delta(\xi)\,. \end{cases} \tag{A.44}$$

where $\Delta(\xi) = (1-u\xi) \to u \ll 1$ as $\xi \to \xi_c$, and a is a positive non-universal constant.

The obtained asymptotics can also be easily generalized for the situation when $g(x = 0) \neq 0$. One has to write $g(x;\xi) =$ (the obtained solution) $+g(x = 0;\xi)$, then put it into the equation, obtain the equation for $g(x = 0;\xi)$, and find the asymptotics for $g(x = 0;\xi)$ It is straightforward to check that qualitatively it does not change the above results.

Appendix B. The asymptotic solution for $p = 4$

In the $p = 4$ case the asymptotic solution of the equations (2.10), (2.11) can be obtained as follows.

Redefining the diagonal parameter $\tilde{g}(\xi)$

$$\tilde{g}(\xi) = \frac{\epsilon}{12} + q(\xi) \tag{B.1}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi}g(x) = -12q(\xi)g(x) + 4g^2(x) - 8g(x)\int_0^1 \mathrm{d}y\, g(y) - 4\int_0^x \mathrm{d}y\,(g(x)-g(y))^2 \tag{B.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}q(\xi) = -\epsilon q - 12q^2 + 4\overline{g^2}\,. \tag{B.3}$$

Then, proceeding as in appendix A, instead of equation (A.7) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi}V'(x;\xi) = -8V'(x;\xi)V(x;\xi) - 12V'(x;\xi)q(\xi)\,. \tag{B.4}$$

Integration over x yields

$$\frac{\mathrm{d}}{\mathrm{d}\xi}V(x;\xi) = -4V^2(x;\xi) - 12q(\xi)V(x;\xi) \tag{B.5}$$

(the integration constant is zero, since $V(x=1)\equiv 0$). The solution of this equation for any given function $q(\xi)$ is

$$V(x;\xi) = \frac{V_0(x)\exp\{-12\int_0^\xi \mathrm{d}\eta\, q(\eta)\}}{1+4V_0(x)\int_0^\xi \mathrm{d}\eta\, \exp\{-12\int_0^\eta \mathrm{d}t\, q(t)\}} \tag{B.6}$$

where $V_0(x)\equiv V(x;\xi=0)=\int_x^1 \mathrm{d}y\,(1-y)g_0'(x)$.

Coming back to the function $g(x;\xi)$ we get

$$g(x;\xi) = \int_0^x \mathrm{d}y\, g'(y) + g(x=0;\xi) = -\int_0^x \mathrm{d}y\,\frac{1}{1-y}V'(y) + g(x=0;\xi). \tag{B.7}$$

Using equation (B.6) we find

$$g(x;\xi) = \int_0^x \mathrm{d}y \frac{g_0'(y)\exp\{-12\int_0^\xi \mathrm{d}\eta\, q(\eta)\}}{[1+4\int_y^1 \mathrm{d}z\,(1-z)g_0'(z)\int_0^\xi \mathrm{d}\eta\,\exp\{-12\int_0^\eta \mathrm{d}t\, q(t)\}]^2} + g(x=0;\xi). \tag{B.8}$$

Putting this result back into the original equation (B.2) we get the equation for $g(x=0;\xi)$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}g(x=0;\xi) = -12q(\xi)g(x=0;\xi) - 4g^2(x=0;\xi) - 8g(x=0;\xi)\overline{g}(\xi) \tag{B.9}$$

where

$$\overline{g}(\xi) = \int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y \frac{g_0'(y)\exp\{-12\int_0^\xi \mathrm{d}\eta\, q(\eta)\}}{[1+4\int_y^1 \mathrm{d}z\,(1-z)g_0'(z)\int_0^\xi \mathrm{d}\eta\,\exp\{-12\int_0^\eta \mathrm{d}t\, q(t)\}]^2}. \tag{B.10}$$

Let us assume now that the parameter $q(\xi)$ decays as $\sim\xi^{-s}$ with $s>1$. Then the integral $\int^\xi \mathrm{d}\eta\, q(\eta)$ converges at large ξ, and for the exponent in (B.8) we find that it is equal to a constant of the order of one: $\exp\{-12\int_0^\xi \mathrm{d}\eta\, q(\eta)\} = A$.

Correspondingly, instead of equations (B.8), (B.10) we get

$$g(x;\xi) \simeq \int_0^x \mathrm{d}y \frac{Ag_0'(y)}{[1+4A\xi\int_y^1 \mathrm{d}z\,(1-z)g_0'(z)]^2} + g(x=0;\xi) \tag{B.11}$$

and

$$\begin{aligned}\overline{g}(\xi) &\simeq \int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y \frac{Ag_0'(y)}{[1+4A\xi\int_y^1 \mathrm{d}z(1-z)g_0'(z)]^2}\\ &= \frac{A\overline{g}_0}{1+4A\xi\overline{g}_0}\end{aligned} \tag{B.12}$$

where $\overline{g}_0 \equiv \int_0^1 \mathrm{d}x\, g_0(x) = \int_0^1 \mathrm{d}x\,(1-x)g_0'(x)$.

Simple analysis of the integral in equation (B.11) shows that actually it is the non-zero derivative $g_0'(x)$ near the point $x=1$ which is important in the asymptotic regime. Whatever the function $g_0(x)$ is in the region $(1-x)\gg(\gamma\xi)^{-1/2}$, it is always decaying like ξ^{-2} there, while for $(1-x)\ll(\gamma\xi)^{-1/2}$ the decay is $(\gamma\xi)^{-1/2}$, where $\gamma = g_0'(x=1)$:

$$g(x;\xi) \sim \begin{cases} \xi^{-2} & (1-x)\gg\dfrac{1}{\sqrt{\gamma\xi}} \\ \dfrac{1}{\sqrt{\gamma\xi}} & (1-x)\ll\dfrac{1}{\sqrt{\gamma\xi}}. \end{cases} \tag{B.13}$$

Besides, using (B.12), from equation (B.9) one finds that $g(x=0;\xi) \sim \xi^{-2}$.

Note now that according to the above asymptotic behaviour of the function $g(x;\xi)$ at scales $\xi \gg 1/u$ the leading contribution to the quantity $\overline{g^2} \equiv \int_0^1 g^2(x;\xi)$ comes from the region $(1-x) \ll \frac{1}{\sqrt{u\xi}}$:

$$\overline{g^2} \sim \frac{1}{\sqrt{\xi}}\left(\frac{1}{\sqrt{\xi}}\right)^2 = \xi^{-3/2}. \tag{B.14}$$

Then, coming back to equation (B.3) we find

$$q(\xi) \simeq \exp(-\epsilon\xi)\int^{\xi} \mathrm{d}t\, t^{-3/2}\exp(+\epsilon t) \sim a_1\xi^{-3/2} + a_2\xi^{-5/2} + \cdots \sim \xi^{-3/2} \tag{B.15}$$

which is self-consistent with the assumption $q(\xi) \sim \xi^{-s}$ (with $s > 1$) made above.

Therefore, the asymptotic behaviour of the solution for $p = 4$ at scales $\xi \gg 1/u$ is given by equation (B.13).

References

[1] Dotsenko V S, Harris B, Sherrington D and Stinchcombe R 1994 *Preprint* cond-mat/9412106 (*J. Phys. A: Math. Gen.* to be published)

[2] Patashinskii A Z and Pokrovskii V L 1979 *Fluctuation Theory of Phase Transitions* (New York: Pergamon)
Parisi G 1988 *Statistical Field Theory* (Reading, MA: Addison-Wesley)

[3] Harris A B 1974 *J. Phys. C: Solid State Phys.* **7** 1671

[4] Lubensky T C and Harris A B 1974 *AIP Conf. Proc.* **24** 311
Grinstein G 1974 *AIP Conf. Proc.* **24** 313
Harris A B and Lubensky T C 1974 *Phys. Rev. Lett.* **33** 1540
Grinstein G and Luther A 1976 *Phys. Rev.* B **13** 1329

[5] Khmelnitskii D E 1975 *Zh. Eksp. Teor. Fiz.* **68** 1960

[6] Mezard M, Parisi G and Virasoro M 1987 *Spin-Glass Theory and Beyond* (Singapore: World Scientific)

[7] Mezard M and Parisi G 1991 *J. Physique I* **1** 809

[8] Mezard M and Young A P 1992 *Europhys. Lett.* **18** 653

[9] Korshunov S 1993 *Phys. Rev.* B **48** 3969

[10] Le Doussal P and Giamarchi T 1995 *Phys. Rev. Lett.* **74** 606

[11] Parisi G 1984 *Proc. Les Houches 1982 (Session XXXIX)* ed J B Zuber and R Stora (Amsterdam: North-Holland)

[12] Polyakov A M 1987 *Gauge Fields and Strings* (New York: Harwood Academic)

[13] Griffiths R 1969 *Phys. Rev. Lett.* **23** 17
Ogielski A T 1985 *Phys. Rev.* B **32** 7384
Bray A J 1987 *Phys. Rev. Lett.* **59** 586

[14] Dotsenko V S 1994 *Introduction to the Theory of Spin-Glasses and Neural Networks* (Singapore: World Scientific)

[15] Gross D, Kanter I and Sompolinsky H 1985 *Phys. Rev. Lett.* **55** 304

[16] Parisi G 1980 *J. Phys. A: Math. Gen.* **13** L115